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Spin Zero Quantum Relativistic Particles in Einstein Universe

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We find exact eigenvalues and eigenfunctions of relativistic massless scalar particle conformally coupled to a background Einstein universe.

KEY WORDS: relativistic particles; Einstein universe.

1. INTRODUCTION

One of the most interesting problems in theoretical physics is the connection between the quantum mechanics and gravity. Since the birth of quantum mechanics there have been many works in this topic (Ahluwalia and Burgard, 1996; Alimohammadi and Shariati, 2000; Alimohammadi and Vakili, 2003; Cardall and Fuller, 1997; Chakrabarti, 1984; Chandrasekhar, 1976; Khorrami *et al.*, 2003; Landau and Lifshitz, 1977; Mukhopadhyay and Chakrabarti, 1999; Page, 1976; Semiz, 1992). Chandrasekhar studied the Dirac equation in a Kerr space–time background. He separated the Dirac equation into radial and angular parts (Chandrasekhar, 1976). Page extended this work to the Dirac equation for an electron around a Kerr–Newman back hole background, in this case also the Dirac equation is separated into decoupled ordinary differential equation (Page, 1976). The separation of variables for the massive complex Dirac equation in the gravitational background of the Dyon black hole has been done in Semiz (1992). Also, in Mukhopadhyay and Chakrabarti (1999) the radial part of Dirac equation in a Schwarzschild geometry has been solved by using WKB approximation method. Besides theoretical works several experiments have been performed to test the theoretical predications, for example in Colella *et al.* (1975) the authors have used a neutron interferometer to observe the quantum mechanical phase shift of neutrons caused by their interaction with the gravitational field of the Earth. Furthermore,

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Nesvizhevsky *et al.* (2002) have measured the quantum energy levels of neutrons in the Earth's gravitational field. Here we shall consider the eigenvalues and eigenfunctions of relativistic massless scalar particle which is conformally coupled to the background of the Einstein universe. We will find the exact eigenvalues and eigenfunctions.

2. MASSLESS CONFORMALLY COUPLED SPIN ZERO QUANTUM RELATIVISTIC PARTICLES

The simplest cases in which an analysis of quantum behavior of particles can be done is the massless spin zero particles in the static space–time. Here we consider the Einstein universe, with scalar particles conformally coupled to it. The line element for this space is given by

$$ds^2 = dt^2 - a^2 d\chi^2 + \sin(\chi)^2 (d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{1}$$

in this case the scalar curvature is

$$R = \frac{6}{a^2}. \tag{2}$$

The equation of motion for massless scalar particles is

$$(\Box + \xi R)u(x) = 0, \tag{3}$$

where $\Box$ is given by

$$\Box = (-g)^{\frac{1}{2}} \partial_\mu [(-g)^{\frac{1}{2}} g^{\mu\nu} \partial_\nu]. \tag{4}$$

The modes $u(x)$ are separated as

$$u_k(x) = \frac{1}{a} y_k(x) \chi_k(t), \tag{5}$$

with $x = (\chi, \theta, \varphi)$ and $y_k(x)$ are solutions of

$$\Delta^3 y_k(x) = -(k^2 - 1) y_k(x), \tag{6}$$

with this separation χ_k satisfying the following equation:

$$\frac{d^2\chi_k}{d\eta^2} + [k^2 + c(\eta)[\xi - \xi(4)]R(\eta)]\chi_k = 0, \tag{7}$$

with $\xi(4) = \frac{1}{6}$, and $c(\eta) = a^2(t)$, where the conformal time parameter η is given by

$$\eta = \int^t a^{-1}(t')dt'. \tag{8}$$

Using the scalar curvature in Eq. (2), the normalized solution of Eq. (1) can be written as

$$\chi_k(\eta) = (2\omega_k)^{-1/2} e^{-i\omega_k \eta}, \tag{9}$$

where

$$\omega_k^2 = k^2 + (6\xi - 1). \tag{10}$$

For conformally coupled case $\xi = \frac{1}{6}$ and we have

$$\omega_k^2 = k^2, \quad k = 1, 2, \ldots \tag{11}$$

$\omega_k = k$ are eigenvalues of massless scalar particles in our interest background. Therefore the eigenvalues are equally spaced. The eigenfunctions y_k of the three-dimensional Laplacian are (Birrell and Davies, 1982; Parker and Fulling, 1974)

$$y_k(x) = \prod\nolimits_{Kj}(\chi) Y_j^m(\theta, \varphi), \quad K = (k, j, M) \tag{12}$$

where

$$M = -j, -j+1, \ldots, j; \quad j = 0, 1, \ldots, k-1, \quad k = 1, 2, \ldots \tag{13}$$

The Y_j^M are spherical harmonics. The functions $\prod_{K_j}^{(+)}(\chi)$ are defined by (Lifshitz and Khalatnikov, 1963)

$$\prod\nolimits_{Kj}^{(+)}(\chi) = \left\{\frac{1}{2}\pi(ik)^2[(ik)^2 + 1^2]\cdots[(ik)^2 + j^2]\right\}^{-1/2} \times \sinh^j(i\chi)\left(\frac{d}{d\cosh(i\chi)}\right)^{1+j} \cos(k\chi). \tag{14}$$

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